April 2, 2019 Add Us On Google

Why Isn't 1 a Prime Number?

And how long has it been a number?

BY EVELYN LAMB

The Sciences

This article was published in Scientific American's former blog network and reflects the views of the author, not necessarily those of Scientific American

An engineer friend of mine recently surprised me by saying he wasn't sure whether the number 1 was prime or not. I was surprised because among mathematicians, 1 is universally regarded as non-prime.

The confusion begins with this definition a person might give of "prime": *a prime number is a positive whole number that is only divisible by 1 and itself.* The number 1 is divisible by 1, and it's divisible by itself. But *itself* and *1* are not two distinct factors. Is 1 prime or not? When I write the definition of prime in an article, I try to remove that ambiguity by saying a prime number has exactly two distinct factors, 1 and itself, or that a prime is a whole number greater than 1 that is only divisible by 1 and itself. But why go to those lengths to exclude 1?

My mathematical training taught me that the good reason for 1 not being considered prime is the fundamental theorem of arithmetic, which states that every number can be written as a product of primes in exactly one way. If 1 were prime, we would lose that uniqueness. We could write 2 as 1×2, or 1×1×2, or 1^{594827}×2. Excluding 1 from the primes smooths that out.

My original plan of how this article would go was that I would explain the fundamental theorem of arithmetic and be done with it. But it's really not so hard to modify the statement of the fundamental theorem of arithmetic to address the 1 problem, and after all, my friend's question piqued my curiosity: how did mathematicians coalesce on this definition of prime? A cursory glance around some Wikipedia pages related to number theory turns up the assertion that 1 used to be considered prime but isn't anymore. But a paper by Chris Caldwell and Yeng Xiong shows the history of the concept is a bit more complicated. I appreciated this sentiment from the beginning of their article: "First, whether or not a number (especially unity) is a prime is a matter of definition, so a matter of choice, context and tradition, not a matter of proof. Yet definitions are not made at random; these choices are bound by our usage of mathematics and, especially in this case, by our notation."

Caldwell and Xiong start with classical Greek mathematicians. They did not consider 1 to be a number in the same way that 2, 3, 4, and so on are numbers. 1 was considered a unit, and a number was composed of multiple units. For that reason, I couldn't have been prime — it wasn't even a number. Ninth-century Arab mathematician al-Kindi wrote that it was not a number and therefore not even or odd. The view that 1 was the building block for all numbers but not a number itself lasted for centuries.

In 1585, Flemish mathematician Simon Stevin pointed out that when doing arithmetic in base 10, there is no difference between the digit 1 and any other digits. For all intents and purposes, 1 behaves the way any other magnitude does. Though it was not immediate, this observation eventually led mathematicians to treat 1 as a number, just like any other number.

Through the end of the 19th century, some impressive mathematicians considered 1 prime, and some did not. As far as I can tell, it was not a matter that caused strife; for the most popular mathematical questions, the distinction was not terribly important. Caldwell and Xiong cite G. H. Hardy as the last major mathematician to consider 1 to be prime. (He explicitly included it as a prime in the first six editions of *A Course in Pure Mathematics,* which were published between 1908 and 1933. He updated the definition in 1938 to make 2 the smallest prime.)

The article mentions but does not delve into some of the changes in mathematics that helped solidify the definition of prime and excluding 1. Specifically, one important change was the development of sets of numbers beyond the integers that behave somewhat like integers.

In the very most basic example, we can ask whether the number -2 is prime. The question may seem nonsensical, but it can motivate us to put into words the unique role of 1 in the whole numbers. The most unusual aspect of 1 in the whole numbers is that it has a multiplicative inverse that is also an integer. (A multiplicative inverse of the number x is a number that when multiplied by x gives 1. The number 2 has a multiplicative inverse in the set of the rational or real numbers, 1/2: 1/2×2=1, but 1/2 is not an integer.) The number 1 happens to be its own multiplicative inverse. No other positive integer has a multiplicative inverse within the set of integers.* The property of having a multiplicative inverse is called being a ***unit***. The number -1 is also a unit within the set of integers: again, it is its own multiplicative inverse. We don't consider units to be either prime or composite because you can multiply them by certain other units without changing much. We can then think of the number -2 as not so different from 2; from the point of view of multiplication, -2 is just 2 times a unit. If 2 is prime, -2 should be as well.

I assiduously avoided defining *prime* in the previous paragraph because of an unfortunate fact about the definition of prime when it comes to these larger sets of numbers: it is wrong! Well, it's not *wrong*, but it is a bit counterintuitive, and if I were the queen of number theory, I would not have chosen for the term to have the definition it does. In the positive whole numbers, each prime number p has two properties:

The number p cannot be written as the product of two whole numbers, neither of which is a unit.

Whenever a product m×n is divisible by p, then m or n must be divisible by p. (To check out what this property means on an example, imagine that m=10, n=6, and p=3.)

The first of these properties is what we might think of as a way to characterize prime numbers, but unfortunately the term for that property is ***irreducible***. The second property is called ***prime***. In the case of positive integers, of course, the same numbers satisfy both properties. But that isn't true for every interesting set of numbers.

As an example, let's look at the set of numbers of the form $a+b\sqrt{-5}$, or $a+ib\sqrt{5}$, where a and b are both integers and i is the square root of -1. If you multiply the numbers 1+√-5 and 1-√-5, you get 6. Of course, you also get 6 if you multiply 2 and 3, which are in this set of numbers as well, with b=0. Each of the numbers 2, 3, 1+√-5, and 1-√-5 cannot be broken down further and written as the product of numbers that are not units. (If you don't take my word for it, it's not too difficult to convince yourself.) But the product (1+√-5)(1-√-5) is divisible by 2, and 2 does not divide either 1+√-5 or 1-√-5. (Once again, you can prove it to yourself if you don't believe me.) So 2 is irreducible, but it is not prime. In this set of numbers, 6 can be factored into irreducible numbers in two different ways.

The number set above, which mathematicians might call $\mathbb{Z}[\sqrt{-5}]$ (pronounced "zee adjoin the square root of negative five" or "zed adjoin the square root of negative five, pip pip, cheerio" depending on what you like to call the last letter of the alphabet), has two units, 1 and -1. But there are similar number sets that have an infinite number of units. As sets like this became objects of study, it makes sense that the definitions of unit, irreducible, and prime would need to be carefully delineated. In particular, if there are number sets with an infinite number of units, it gets more difficult to figure out what we mean by unique factorization of numbers unless we clarify that units cannot be prime. While I am not a math historian or a number theorist and would love to read more about exactly how this process took place before speculating further, I think this is one development Caldwell and Xiong allude to that motivated the exclusion of 1 from the primes.

As happens so often, my initial neat and tidy answer for why things are the way they are ended up being only part of the story. Thanks to my friend for asking the question and helping me learn more about the messy history of primality.

*This sentence was edited after publication to clarify that no other positive integer has a multiplicative inverse that is also an integer.

EVELYN LAMB is a freelance math and science writer based in Salt Lake City, Utah.

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